

Baire categories and classes of analytic functions in which the Wiman-Valiron type inequality can be almost surely improved

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Abstract

Let $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ ($z \in \mathbb{C}$) be an analytic function in the unit disk and f_t be an analytic function of the form $f_t(z) = \sum_{n=0}^{+\infty} a_n e^{i\theta_n t} z^n$, where $t \in \mathbb{R}$, $\theta_n \in \mathbb{N}$, and h be a positive continuous function on $(0, 1)$ increasing to $+\infty$ and such that $\int_{r_0}^1 h(r) dr = +\infty$, $r_0 \in (0, 1)$. If the sequence $(\theta_n)_{n \geq 0}$ satisfies the inequality

$$\overline{\lim}_{n \rightarrow +\infty} \frac{1}{\ln n} \ln \frac{\theta_n}{\theta_{n+1} - \theta_n} \leq \delta \in [0, 1/2),$$

then for all analytic functions f_t almost surely for t there exists a set $E = E(\delta, t) \subset (0, 1)$ such that $\int_E h(r) dr < +\infty$ and

$$\overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \frac{\ln M_f(r, t) - \ln \mu_f(r)}{2 \ln h(r) + \ln \ln \{h(r) \mu_f(r)\}} \leq \frac{1 + 2\delta}{4 + 3\delta},$$

where $M_f(r, t) = \max\{|f_t(z)| : |z| = r\}$, $\mu_f(r) = \max\{|a_n| r^n : n \geq 0\}$ for $r \in [0, 1)$.

Subject Classification: 30B10, 30B20, 54E52

Keywords: random analytic functions, Wiman-Valiron's type inequality, Baire categories

1 Introduction

Let H be the class of positive continuous functions on the interval $(0, 1)$ increasing to $+\infty$ and such that $\int_{r_0}^1 h(r)dr = +\infty$, $r_0 \in (0, 1)$.

For a measurable set $E \subset (0, 1)$, the h -measure of E is defined by

$$h\text{-meas}(E) \stackrel{\text{def}}{=} \int_E h(r)dr,$$

where $h \in H$. It is clear that $h\text{-meas}((0, 1)) = +\infty$.

Let f be an analytic function in the unit disc $\mathbb{D} = \{z: |z| < 1\}$ of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n. \quad (1)$$

For $r \in (0, 1)$ we denote the maximum modulus of the function f by $M_f(r) = \max\{|f(z)|: |z| = r\}$, and the maximal term of the series (1) by $\mu_f(r) = \max\{|a_n|r^n: n \geq 0\}$. Let also

$$G_f(r) = \sum_{n=0}^{+\infty} |a_n|r^n, \quad S_f(r) = \left(\sum_{n=0}^{+\infty} |a_n|^2 r^{2n} \right)^{1/2},$$

$$\Delta_h(r, f) = \frac{\ln M_f(r) - \ln \mu_f(r)}{2 \ln h(r) + \ln_2 \{h(r)\mu_f(r)\}},$$

$$E(\eta, f, h) = \{r \in (0, 1): M_f(r) > \mu_f(r)(h^2(r) \ln \{h(r)\mu_f(r)\})^\eta\},$$

where $\ln_k x \stackrel{\text{def}}{=} \ln(\ln_{k-1} x)$ ($k \geq 2$), $\ln_1 x \stackrel{\text{def}}{=} \ln x$.

From the results proved in [1] it follows that in the case when $h(r) = (1 - r)^{-1}$, for every analytic function f in \mathbb{D} of the form (1) there exists a set $E \subset (0, 1)$ of finite logarithmic measure, i.e. $h\text{-meas}(E) < +\infty$, such that

$$\overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f) \leq \frac{1}{2}. \quad (2)$$

In [2] the similar statement is proved with an arbitrary function $h \in H$ for which either $\ln h(r) = O(\ln_2 G_f(r))$ or $\ln_2 G_f(r) = O(\ln h(r))$ ($r \rightarrow 1 - 0$).

In [3] it is noted that the constant $1/2$ in the inequality (2) cannot be replaced by a smaller number in general. Indeed, if $g(z) = \sum_{n=1}^{+\infty} \exp\{\sqrt{n}\}z^n$, then for $h(r) = (1 - r)^{-1}$ we have

$$\lim_{r \rightarrow 1-0} \frac{M_g(r)}{h(r)\mu_g(r) \ln^{1/2}\{\mu_g(r)h(r)\}} \geq C > 0.$$

In connection with this the following question arises naturally: *how can one describe the “quantity” of those analytic functions for which inequality (2) can be improved?*

In the paper [4] it is proved that in some probability sense for “majority” of analytic functions the constant $1/2$ in the inequality (2) can be replaced by $1/4$. Similar statement is proved in [2] in reference to the inequality (2) with any function $h \in H$ described above.

At the same time, the classes of random analytic functions considered in [2], [4] do not include all analytic functions of the form

$$f_t(z) = \sum_{n=0}^{+\infty} a_n e^{i\theta_n t} z^n, \quad (3)$$

where $(\theta_n)_{n \geq 0}$ is an arbitrary sequence of nonnegative integers. Note that $f_0(z) \equiv f(z)$.

We suppose that the sequence $(\theta_n)_{n \geq 0}$ satisfies the inequality

$$\frac{\theta_{n+1}}{\theta_n} \geq q > 1 \quad (n \geq 0). \quad (4)$$

In the case of $q \geq 2$ analytic functions of the form (4) satisfies the conditions of theorems from [2], [4] mentioned above.

We also remark that the possibility of improvement of Wiman-Valiron's inequality for entire functions of the form (3) was considered earlier by M. Still [5] and P. Filevych [6] (see also [7]). A similar question for the class of entire functions of two variables was considered in the papers [8], [9] and [10]. In [11] the “quantity” of those entire functions for which classical Wiman-Valiron's inequality can be improved, is described in the sense of Baire's categories.

Here we consider the formulated *question* in the class of analytic functions in \mathbb{D} of the form (3). The proved theorems complement in this case theorems from [4, 2] and are analogues of the statements from [5] and [11].

2 Auxiliary lemmas

We need Lemma 2 from [5] (see also [6]).

Lemma 2.1 ([5]). *If a sequence $(\theta_n)_{n \geq 0}$ satisfies the condition (4), then for all sequences $(a_n)_{n \geq 0}$, $a_n \in \mathbb{C}$, and all $\beta > 0$, $N \geq 0$ we have*

$$P_0 \left(\left\{ t \in [0, 2\pi]: \max_{0 \leq \psi \leq 2\pi} \left| \sum_{k=0}^N a_k e^{ik\psi} e^{i\theta_k t} \right| \geq A_{\beta q} S_N \ln^{1/2} N \right\} \right) \leq \frac{1}{N^\beta},$$

where $A_{\beta q}$ is a constant which depends only on β and q , $S_N = \sum_{n=0}^N |a_n|^2$, $P_0 = \frac{\mathfrak{m}}{2\pi}$, \mathfrak{m} is the Lebesgue measure on the real line.

Lemma 2.2 ([4]). *Let $k(r)$ be a continuous increasing to $+\infty$ function on $(0, 1)$, $E \subset (0, 1)$ be an open set such that there exists a sequence $0 < p_1 \leq \dots \leq p_n \rightarrow 1$ ($n \rightarrow +\infty$) outside E . Then there exists a sequence $0 < r_1 \leq \dots \leq r_n \rightarrow 1$ ($n \rightarrow +\infty$) such that for all $n \in \mathbb{N}$*

$$1) \ r_n \notin E,$$

$$2) \ \ln k(r_n) \geq \frac{n}{2},$$

$$3) \ \text{if } (r_n; r_{n+1}) \cap E \neq (r_n, r_{n+1}), \text{ then } k(r_{n+1}) \leq ek(r_n).$$

Lemma 2.3 ([2]). *Let $\varphi_1(x)$ and $\varphi_2(x)$ be positive continuous increasing to $+\infty$ functions on $[0, +\infty)$ such that $\int_0^{+\infty} \frac{dx}{\varphi_i(x)} < +\infty$ ($i \in \{1, 2\}$), $h \in H$ and $g_1(x) = \ln G_f(e^x)$ ($x < 0$). Then there exists a set $E \subset (0, 1)$ such that $h\text{-meas}(E) < +\infty$ and for all $r \in (0, 1) \setminus E$ we get*

$$g_1''(\ln r) \leq h(r)\varphi_2(h(r)\varphi_1(g_1(\ln r))).$$

We also denote

$$A(r) = g_1'(\ln r) = \frac{d \ln G_f(r)}{d \ln r} = \sum_{n=0}^{+\infty} \frac{n|a_n|r^n}{G_f(r)},$$

$$B^2(r) = g_1''(\ln r) = \sum_{n=0}^{+\infty} \frac{n^2|a_n|r^n}{G_f(r)} - A^2(r).$$

Lemma 2.4. *For $h \in H$ and all $\varepsilon > 0$ there exists a set $E \subset (0, 1)$ such that $h\text{-meas}(E) < +\infty$ and for all $r \in (0, 1) \setminus E$ we have*

$$A(r) \leq h(r) \ln\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\},$$

$$B^2(r) \leq h^{2+\varepsilon}(r) \ln\{h(r)\mu_f(r)\} \ln_2^{2+\varepsilon}\{h(r)\mu_f(r)\}.$$

Proof. Let (Ω, \mathcal{A}, P) be a probability space which contains the discrete random variable ξ with the distribution

$$P(\xi = n) = \frac{|a_n|e^{nx}}{G_f(e^x)}.$$

Then the mean $M\xi = g_1'(x)$ and the variance $D\xi = g_1''(x)$.

Let $x = \ln r < 0$. Using Chebyshev's inequality we get $P(|\xi - g_1'(x)| < \sqrt{2g_1''(x)}) \geq 1/2$, i.e.

$$\begin{aligned} g(x) &\leq 2 \sum_{|n - g_1'(x)| < \sqrt{2g_1''(x)}} |a_n|e^{xn} \leq \\ &\leq 2\mu_f(r) \sum_{|n - g_1'(x)| < \sqrt{2g_1''(x)}} 1 \leq 2\mu_f(r)(2\sqrt{2g_1''(x)} + 1). \end{aligned} \quad (5)$$

For fixed $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ we define

$$\begin{aligned} E_1 &= \{x < 0: g_1''(x) > h(e^x)g_1'(x)(\ln g_1'(x))^{1+\varepsilon_1}, g_1'(x) \geq 2\}, \\ E_2 &= \{x < 0: g_1'(x) > h(e^x)g_1(x)(\ln g_1(x))^{1+\varepsilon_2}, g_1(x) \geq 2\}. \end{aligned}$$

So,

$$\int_{E_1 \cup E_2} h(e^x) dx = \int_E \frac{h(r)}{r} dr < +\infty, \quad \int_E h(r) dr < +\infty,$$

where E is the image of the set $E_1 \cup E_2$ by the mapping $r = e^x$. Therefore, $h\text{-meas} E = \int_E h(r) dr < +\infty$.

Then from (5) we obtain as $r \rightarrow 1 - 0$, ($r \notin E$)

$$\begin{aligned} g(\ln r) &\leq 2\mu_f(r) \left(2\sqrt{2} \sqrt{h(e^x)g_1'(x) \ln^{1+\varepsilon_1} g_1'(x) + 1} \right) \leq \\ &\leq 4\mu_f(r) \left(\sqrt{2h^2(e^x)g_1(x) \ln^{1+\varepsilon_2} g_1(x) \ln^{\frac{1+\varepsilon_1}{2}} \{h(e^x)g_1(x) \ln^{1+\varepsilon_2} g_1(x)\} + 1} \right) \leq \\ &\leq 6\mu_f(r)h(r) \sqrt{g_1(x) \ln^{\frac{1+\varepsilon_2}{2}} g_1(x) \ln^{\frac{1}{2}+\varepsilon_1} \{h(r)g_1(x)\}}, \\ g_1(x) = \ln g(x) &\leq \ln 6 + \ln \{h(r)\mu_f(r)\} + \ln g_1(x) + \ln_2 \{h(r)\mu_f(r)\}, \\ g_1(x) &\leq 2 \ln \{h(r)\mu_f(r)\}. \end{aligned}$$

Now for $\delta > 2(\varepsilon_1 + \varepsilon_2)$ we have

$$\begin{aligned} G_f(r) &\leq \mu_f(r)h(r) \ln^{1/2} \{h(r)\mu_f(r)\} \times \\ &\times \left(\ln_2 \{h(r)\mu_f(r)\} \ln \{h(r) \ln \{h(r)\mu_f(r)\}\} \right)^{\frac{1+\delta}{2}}, \\ M_f(r) &\leq G_f(r) \leq \mu_f(r)h(r) \ln^{1/2} \{h(r)\mu_f(r)\} \ln^{1/2+\delta} h(r) \ln_2^{1+\delta} \{h(r)\mu_f(r)\}, \quad (6) \\ g_1(x) &= (1 + o(1)) \ln \{h(r)\mu_f(r)\}, \quad r \rightarrow 1 - 0, \quad (r \notin E). \end{aligned}$$

If we choose $\varphi_i(x) = (x+2) \ln^{1+\varepsilon_0/2}(2+x)$, $i \in \{1, 2\}$ in Lemma 2.3, then we get that outside a set of finite h -measure

$$\begin{aligned} A(r) &\leq h(r)\varphi(g_1(\ln r)) \leq h(r)g_1(\ln r) \ln^{1+\varepsilon_0} g_1(\ln r) \leq \\ &\leq h(r) \ln \{h(r)\mu_f(r)\} \ln^{1+\varepsilon} \{h(r)\mu_f(r)\}. \\ B^2(r) &\leq h(r)\varphi_2(h(r)\varphi_1(g_1(\ln r))) \leq \\ &\leq h(r)h(r)\varphi_1(g_1(\ln r)) \ln^{1+\varepsilon_0} (h(r)\varphi_1(g_1(\ln r))) \leq \\ &\leq h^2(r)g_1(\ln r) \ln^{1+\varepsilon_0} g_1(\ln r) \ln^{1+\varepsilon_0} \{h(r)g_1(\ln r) \ln^{1+\varepsilon_0} g_1(\ln r)\} \leq \\ &\leq h^{2+\varepsilon}(r) \ln^{1+\varepsilon} \{h(r)\mu_f(r)\}. \end{aligned}$$

□

3 Classes of analytic functions in which the Wiman-Valiron type inequality (2) can be almost surely improved

In the sequel, the notion “almost surely” will be used in the sense that the corresponding property holds almost everywhere with respect to Lebesgue measure on the real line. Here we prove the following theorem.

Theorem 3.1. *If $f(z, t)$ is an analytic function of the form (3) and a sequence $(\theta_n)_{n \geq 0}$ satisfies condition (4), then for all $\delta > 0$ and almost surely for t there exists a set $E(\delta, t) \subset (0, 1)$ such that $h\text{-meas}(E(\delta, t)) < +\infty$ and the maximum modulus $M_f(r, t) = M_{f_t}(r) = \max_{|z| \leq r} |f_t(z)|$ satisfies the inequality*

$$M_f(r, t) \leq \mu_f(r) \sqrt{h(r)} \ln^{1/4} \{h(r) \mu_f(r)\} \ln^{3/4+\delta} h(r) \ln_2^{1+\delta} \{h(r) \mu_f(r)\} \quad (7)$$

for $r \in (0, 1) \setminus E(\delta, t)$.

We note that from inequality (7) it follows that

$$\overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) = \overline{\lim}_{r \rightarrow 1-0} \frac{\ln M_f(r, t) - \ln \mu_f(r)}{2 \ln h(r) + \ln_2 \{h(r) \mu_f(r)\}} \leq \frac{1}{4}. \quad (8)$$

Proof. Let (Ω, \mathcal{A}, P) be a probability space which contains a random variable $X = X(\omega): \Omega \rightarrow \mathbb{Z}_+$ with the distribution $P(X = n) = |a_n| r^n / G_f(r)$. Using Markov's inequality for the random variable X with mean value $MX = A(r)$ we get

$$\sum_{n \geq C} \frac{|a_n| r^n}{G_f(r)} = P(X \geq C) \leq \frac{MX}{C} = \frac{A(r)}{C}.$$

Let $C = C(r) = A(r) h(r) \ln^{1/2+\delta} \{h(r) \mu_f(r)\}$ and

$$C_1(r) = h^2(r) \ln^2 \{h(r) \mu_f(r)\}.$$

By Lemma 2.4 $C_1(r) > C(r)$ for $r \in (r_0, 1) \setminus E$. Using (6) we have

$$\begin{aligned} \sum_{n \geq C_1(r)} |a_n| r^n &\leq \sum_{n \geq C(r)} |a_n| r^n \leq \frac{A(r) G_f(r)}{A(r) h(r) \ln^{1/2+\delta} \{h(r) \mu_f(r)\}} \leq \\ &\leq \frac{h(r) \mu_f(r) \ln^{1/2+\delta} h(r) \ln^{1/2+\delta} \{h(r) \mu_f(r)\}}{h(r) \ln^{1/2+\delta} \{h(r) \mu_f(r)\}} = \mu_f(r) \ln^{1/2+\delta} h(r) \end{aligned} \quad (9)$$

for $r \notin E$, where E is a set of finite h -measure.

We put $k(r) = h(r)\mu_f(r)$ in Lemma 2.2 and let $(r_k)_{k \geq 0}$ be the sequence for which consequences of this lemma are valid. We denote by F_k the set of $t \in \mathbb{R}$ such that

$$W(r_k) = \max_{0 \leq \psi \leq 2\pi} \left| \sum_{n \leq [C_1(r_k)]} a_n r_k^n e^{in\psi} e^{i\theta_n t} \right| \geq A_{\beta q} S_{[C_1(r_k)]}(r_k) \ln^{1/2}[C_1(r_k)].$$

It follows from Lemma 2.2 with $\beta = 2$ that

$$\sum_{k=1}^{+\infty} P(F_k) \leq \sum_{k=1}^{+\infty} \frac{1}{[C_1(r_k)]^2} \leq \sum_{k=1}^{+\infty} \frac{1}{[\ln\{\mu_f(r_k)h(r_k)\}]^2} \leq \sum_{k=1}^{+\infty} \frac{4}{k^2} < +\infty.$$

Then by Borel-Cantelli's lemma for $k \geq k_0(t)$ and almost surely for $t \in \mathbb{R}$ we obtain

$$W(r_k) < A_q S_{[C_1(r_k)]}(r_k) \ln^{1/2}[C_1(r_k)]. \quad (10)$$

From inequalities (6), (10) and $S_{[C_1(r)]}(r) \leq M_f(r)\mu_f(r)$ it follows that

$$\begin{aligned} W(r_k) &< \sqrt{\mu_f(r_k)} \sqrt{\mu_f(r_k)h(r_k)} \ln^{1/4}\{h(r_k)\mu_f(r_k)\} \times \\ &\times \ln^{1/4+2\delta/3} h(r_k) \ln_2^{1/2+2\delta/3}\{h(r_k)\mu_f(r_k)\} \ln^{1/2}(h^2(r) \ln^2\{h(r)\mu_f(r)\}) \leq \\ &\leq \mu_f(r_k) \sqrt{h(r_k)} \ln^{1/4}\{h(r_k)\mu_f(r_k)\} \ln^{3/4+3\delta/4} h(r_k) \ln_2^{1+3\delta/4}\{h(r_k)\mu_f(r_k)\}. \end{aligned} \quad (11)$$

Since

$$M_f(r, t) \leq \sum_{n \geq C_1(r)} |a_n| r^n + W(r),$$

from (9) and (11) we get

$$\begin{aligned} M_f(r_k, f) &\leq \mu_f(r_k) \sqrt{h(r_k)} \ln^{1/4}\{h(r_k)\mu_f(r_k)\} \times \\ &\times \ln^{3/4+4\delta/5} h(r_k) \ln_2^{1+4\delta/5}\{h(r_k)\mu_f(r_k)\}. \end{aligned} \quad (12)$$

We suppose that $r_{k_2(t)} \in (0, 1)$ is some number outside the set E . Then for $r \in (r_p, r_{p+1})$, $p > k_2(t)$ by Lemma 2.2 we obtain

$$\mu_f(r_{p+1})h(r_{p+1}) \leq e\mu_f(r_p)h(r_p) \leq e\mu_f(r)h(r), \quad (13)$$

$$\mu_f(r_{p+1}) = h(r_{p+1}) \frac{\mu_f(r_{p+1})}{h(r_{p+1})} \leq eh(r_p) \frac{\mu_f(r_p)}{h(r_{p+1})} \leq eh(r) \frac{\mu_f(r)}{h(r_{p+1})} \leq e\mu_f(r), \quad (14)$$

$$h(r_{p+1}) = \frac{\mu_f(r_{p+1})h(r_{p+1})}{\mu_f(r_{p+1})} \leq e \frac{\mu_f(r)h(r)}{\mu_f(r_{p+1})} \leq eh(r). \quad (15)$$

Finally, from (12) we have for $r \in (r_p, r_{p+1})$

$$\begin{aligned} M_f(r, t) &\leq M_f(r_{p+1}, t) \leq \\ &\leq \mu_f(r) \sqrt{h(r)} \ln^{1/4}\{h(r)\mu_f(r)\} \ln^{3/4+\delta} h(r) \ln_2^{1+\delta}\{h(r)\mu_f(r)\} \end{aligned}$$

almost surely for $t \in \mathbb{R}$. \square

By \mathcal{L} we denote the class of increasing to $+\infty$ functions $l(x)$ on $[0, +\infty)$. Let

$$\gamma(l) = \overline{\lim}_{x \rightarrow +\infty} \frac{\ln l(x)}{\ln x}.$$

Now we consider the class of analytic functions of the form (3), for which the sequence $(\theta_n)_{n \geq 0}$ satisfies the condition

$$\frac{\theta_{n+1}}{\theta_n} \geq 1 + \frac{1}{\varphi(n)}, \quad \varphi \in \mathcal{L}. \quad (16)$$

What constant can we put in the inequality (8) instead of $1/4$ for this class of analytic functions? Under which conditions on the function $\varphi(x)$ does the inequality (8) hold? We give answers to these questions in Corollaries 3.4 and 3.5.

Firstly, we note that one cannot sharpen inequality (2) for a rapidly growing function $\varphi(x)$. Indeed, if $\varphi(x) = x$, then we may choose $\theta_n = n$, $h(r) = (1-r)^{-1}$ and $g(z) = \sum_{n=0}^{+\infty} e^{\sqrt{n}} z^n$. As it is known from [3],

$$\begin{aligned} M_g(r, t) &= \max\{|g(r, t)| : |z| \leq r\} = \max_{0 \leq \psi \leq 2\pi} \left| \sum_{n=0}^{+\infty} a_n r^n e^{int} e^{in\psi} \right| = \\ &= \max_{0 \leq \psi \leq 2\pi} \left| \sum_{n=0}^{+\infty} a_n r^n e^{in(t+\psi)} \right| = \max_{0 \leq \psi \leq 2\pi} \left| \sum_{n=0}^{+\infty} a_n r^n e^{in\psi} \right| = \\ &= M_g(r) \geq C_1 \mu_g(r) h(r) \ln^{1/2} \{\mu_g(r) h(r)\}, \end{aligned}$$

when $r \rightarrow 1-0$ and $t \in \mathbb{R}$. So, in order to improve inequality (2) $\varphi(x)$ must satisfy the condition $\gamma(\varphi) < 1$.

Theorem 3.2. *Let $f_t(z)$ be an analytic function of the form (3), $h \in H$, sequence $(\theta_n)_{n \geq 0}$ satisfy condition (16), where $\varphi \in \mathcal{L}$. If $v \in \mathcal{L}$ and $\gamma(v) \leq 1/4$, then almost surely for $t \in \mathbb{R}$, all $\varepsilon > 0$ there exists a set $E(\varepsilon, t) \subset (0, 1)$ such that $h\text{-meas}(E(\varepsilon, t)) < +\infty$ and for $r \in (0, 1) \setminus E(\varepsilon, t)$ we have*

$$\begin{aligned} M_f(r, t) &\leq \sqrt{h(r) \ln h(r)} \mu_f(r) \ln^{1/4} \{h(r) \mu_f(r)\} \ln^{1+\varepsilon} \{\ln h(r) \ln \{h(r) \mu_f(r)\}\} \times \\ &\times \left(v \left(8h^2(r) \ln \{h(r) \mu_f(r)\} \right) + \varphi^{\frac{1}{2}} \left(\frac{h^{\frac{3}{2}}(r) \ln^{\frac{5}{4}} \{h(r) \mu_f(r)\} \ln_2^{1+\varepsilon} \{h(r) \mu_f(r)\}}{v(h(r) \ln \{h(r) \mu_f(r)\})} \right) \right). \end{aligned} \quad (17)$$

In order to prove this theorem we need a lemma from [6].

Lemma 3.3 ([6]). *If $(\theta_n)_{n \geq 0}$ satisfies condition (16), then for all $\beta > 0$*

$$P \left(\max_{0 \leq \psi \leq 2\pi} \left| \sum_{k=1}^N a_k e^{ik\psi} e^{i\theta_k t} \right| \geq A_\beta \left\{ \varphi(N) S_N \ln N \right\}^{1/2} \right) \leq \frac{1}{N^\beta},$$

where A_β is a constant which depends only on β .

Proof of Theorem 3.2. By Lemma 2.4 we obtain outside a set of finite h -measure

$$A(r) \leq h(r) \ln\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}. \quad (18)$$

We put $C(r) = A(r)T(r)$, where

$$T(r) = \frac{\sqrt{h(r)} \ln^{1/4}\{h(r)\mu_f(r)\}}{v(h^2(r) \ln\{h(r)\mu_f(r)\})}.$$

Then from (18) we have

$$\begin{aligned} C(r) &= A(r)T(r) \leq h(r) \ln\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}T(r) = \\ &= \frac{h^{3/2}(r) \ln^{5/4}\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{v(h^2(r) \ln\{h(r)\mu_f(r)\})} = C_1(r). \end{aligned}$$

Now using Markov's inequality we get

$$\begin{aligned} \sum_{n \geq C_1(r)} |a_n| r^n &\leq \sum_{n \geq C(r)} |a_n| r^n \leq \frac{G_f(r)}{T(r)} \leq \\ &\leq \frac{h(r)\mu_f(r) \{\ln h(r) \ln\{h(r)\mu_f(r)\}\}^{1/2} \ln^{1+\delta}\{\ln h(r) \ln\{h(r)\mu_f(r)\}\}}{\sqrt{h(r)} \ln^{1/4}\{h(r)\mu_f(r)\}} \times \\ &\quad \times v(h^2(r) \ln\{h(r)\mu_f(r)\}) = \\ &= \mu_f(r) \sqrt{h(r) \ln h(r)} \ln^{1/4}\{h(r)\mu_f(r)\} \ln^{1+\delta}\{\ln h(r) \ln\{h(r)\mu_f(r)\}\} \times \\ &\quad \times v(h^2(r) \ln\{h(r)\mu_f(r)\}). \end{aligned} \quad (19)$$

Let $k(r) = h(r)\mu_f(r)$ and $(r_k)_{k \geq 0}$ be the sequence for which consequences of Lemma 2.2 are valid. Denote by G_k the set of such $t \in \mathbb{R}$, for which

$$\begin{aligned} W_1(r_k) &= \max_{0 \leq \psi \leq 2\pi} \left| \sum_{n \leq [C_1(r_k)]} a_n r_k^n e^{in\psi} e^{i\theta_n t} \right| \geq \\ &\geq A_\beta \left(\varphi([C_1(r_k)]) S_{[C_1(r_k)]}(r_k) \ln[C_1(r_k)] \right)^{1/2}, \end{aligned}$$

where $S_f^2(r) = \sum_{n=0}^{+\infty} |a_n|^2 r^{2n}$.

Since $\gamma(v) \leq 1/4$, we have

$$\begin{aligned} C_1(r) &> \frac{h^{3/2}(r) \ln^{5/4}\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{(h^2(r) \ln\{h(r)\mu_f(r)\})^{1/4}} > \\ &> h(r) \ln\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\} > \ln\{h(r)\mu_f(r)\}. \end{aligned}$$

So, by Lemma 2.2 $\ln k(r_n) > n/2$, i.e. $\ln\{h(r_n)\mu_f(r_n)\} > n/2$. Then

$$C_1(r_n) > \ln\{h(r_n)\mu_f(r_n)\} > n/2.$$

Using Lemma 3.3 with $\beta = 2$ we get

$$\sum_{k=1}^{+\infty} P(G_k) < \sum_{k=1}^{+\infty} \frac{1}{N^\beta(r_k)} < \sum_{k=1}^{+\infty} \frac{4}{k^2} < +\infty.$$

Now by Borel-Cantelli's lemma for $k \geq k_2(t)$ and almost surely $t \in \mathbb{R}$ we obtain

$$W_1(r_k) < A_\beta \left(\varphi([C_1(r_k)]) S_{[C_1(r_k)]}(r_k) \ln[C_1(r_k)] \right)^{1/2}.$$

Using the inequality $S_f^2(r) \leq G_f(r) \mu_f(r)$, we obtain

$$\begin{aligned} W_1(r_k) &< \sqrt{h(r_k) \ln h(r_k)} \mu_f(r_k) \ln^{1/4} \{h(r_k) \mu_f(r_k)\} \times \\ &\quad \times \ln^{1+\delta} \{ \ln h(r_k) \ln \{h(r_k) \mu_f(r_k)\} \} \times \\ &\quad \times \varphi^{1/2} \left(\frac{h^{3/2}(r_k) \ln^{5/4} \{h(r_k) \mu_f(r_k)\} \ln_2^{1+\delta} \{h(r_k) \mu_f(r_k)\}}{v(h^2(r_k) \ln \{h(r_k) \mu_f(r_k)\})} \right). \end{aligned} \quad (20)$$

It follows from (20) and (17) that

$$\begin{aligned} M_f(r_k, t) &\leq \sqrt{h(r_k) \ln h(r_k)} \mu_f(r_k) \ln^{1/4} \{h(r_k) \mu_f(r_k)\} \times \\ &\quad \times \ln^{1+\delta} \{ \ln h(r_k) \ln \{h(r_k) \mu_f(r_k)\} \} \left(v(h^2(r_k) \ln \{h(r_k) \mu_f(r_k)\}) + \right. \\ &\quad \left. + \varphi^{1/2} \left(\frac{h^{3/2}(r_k) \ln^{5/4} \{h(r_k) \mu_f(r_k)\} \ln_2^{1+\delta} \{h(r_k) \mu_f(r_k)\}}{v(h^2(r_k) \ln \{h(r_k) \mu_f(r_k)\})} \right) \right). \end{aligned}$$

Using (13)–(15) we get for $r \in (r_p, r_{p+1})$

$$\begin{aligned} M_f(r, t) &\leq \sqrt{h(r) \ln h(r)} \mu_f(r) \ln^{1/4} \{h(r) \mu_f(r)\} \times \\ &\quad \times \ln^{1+2\delta} \{ \ln h(r) \ln \{h(r) \mu_f(r)\} \} \left(v(8h^2(r) \ln \{h(r) \mu_f(r)\}) + \right. \\ &\quad \left. + \varphi^{1/2} \left(\frac{h^{3/2}(r) \ln^{5/4} \{h(r) \mu_f(r)\} \ln^{1+2\delta} \{h(r) \mu_f(r)\}}{v(h^2(r) \ln \{h(r) \mu_f(r)\})} \right) \right). \end{aligned}$$

□

In the case when $\ln \varphi(x) = o(\ln_2 x)$, $x \rightarrow +\infty$ we have the following corollary.

Corollary 3.4. *Let $f_t(z)$ be an analytic function of the form (3), $h \in H$, a sequence $(\theta_n)_{n \geq 0}$ satisfy condition (16), where $\varphi \in \mathcal{L}$ and $\ln \varphi(x) = O(\ln_2 x)$,*

$x \rightarrow +\infty$. Then there exists a set $E(\delta, t) \subset (0, 1)$ such that $h\text{-meas}(E(\delta, t)) < +\infty$ and almost surely for $t \in \mathbb{R}$ we get

$$\overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) \leq \frac{1}{4}.$$

Corollary 3.5. Let $f_t(z)$ be an analytic function of the form (3), $h \in H$, a sequence $(\theta_n)_{n \geq 0}$ satisfy condition (16), where $\varphi \in \mathcal{L}$ and

$$\gamma(\varphi) = \overline{\lim}_{n \rightarrow +\infty} \frac{1}{\ln n} \ln \frac{\theta_n}{\theta_{n+1} - \theta_n} \leq \delta \in [0, 1/2). \quad (21)$$

Then for all analytic functions f_t there exists a set $E(\delta, t) \subset (0, 1)$ such that $h\text{-meas}(E(\delta, t)) < +\infty$ and almost surely for $t \in \mathbb{R}$ we have

$$\overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) \leq \frac{1 + 3\delta}{4 + 2\delta}.$$

Proof. If $\gamma(\varphi) = \delta \in [0, 1/2)$, then we may choose $v(x) = x^\alpha$, $\alpha \in [0, 1/4)$. So,

$$\begin{aligned} \ln v(8h^2(r) \ln\{h(r)\mu_f(r)\}) &= (\alpha + o(1))(2 \ln h(r) + \ln_2\{h(r)\mu_f(r)\}), \\ \ln \left(\varphi^{1/2} \left(\frac{h^{3/2}(r) \ln^{5/4}\{h(r)\mu_f(r)\} \ln_2^{1+\varepsilon}\{h(r)\mu_f(r)\}}{v(h^2(r) \ln\{h(r)\mu_f(r)\})} \right) \right) &\leq \\ &\leq (1 + o(1)) \left(\frac{3\delta}{4} \ln h(r) + \frac{5\delta}{8} \ln_2\{h(r)\mu_f(r)\} - \delta\alpha \ln h(r) - \frac{\delta\alpha}{2} \ln_2\{h(r)\mu_f(r)\} \right) = \\ &= \left(\frac{3\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) 2 \ln h(r) + \left(\frac{5\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) \ln_2\{h(r)\mu_f(r)\} \leq \\ &\leq \left(\frac{5\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) (2 \ln h(r) + \ln_2\{h(r)\mu_f(r)\}). \end{aligned} \quad (22)$$

From the equation $\alpha = \frac{5\delta}{8} - \frac{\delta\alpha}{2}$ we may determine $\alpha = \frac{5\delta}{4(2+\delta)}$ and get as $r \rightarrow 1 - 0$

$$\begin{aligned} \ln M_f(r, t) &\leq (1 + o(1)) \left(\frac{1}{2} \ln h(r) + \ln \mu_f(r) + \right. \\ &\quad \left. + \frac{1}{4} \ln_2\{h(r)\mu_f(r)\} + \alpha(\ln h(r) + \ln_2\{h(r)\mu_f(r)\}) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) &= \overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \frac{\ln M_f(r, t) - \ln \mu_f(r)}{2 \ln h(r) + \ln_2\{h(r)\mu_f(r)\}} \leq \\ &\leq \frac{1}{4} + \alpha = \frac{1}{4} + \frac{5\delta}{4(2+\delta)} = \frac{1+3\delta}{4+2\delta}. \end{aligned}$$

□

So, we can improve inequality (2) for all analytic functions of the form (3) and all $h \in H$, when $\gamma(\varphi) < 1/2$. As it is noted above, this inequality cannot be improved if $\gamma(\varphi) \geq 1$. Can we improve inequality (2) for all analytic functions of the form (3) by condition $\gamma(\varphi) < 1$?

Corollary 3.6 gives a positive answer to this question by some choice of the function $h(r)$.

Corollary 3.6. *Let $f_t(z)$ be an analytic function of the form (3), $h \in H$: $\ln_2 \mu_f(r) = o(\ln h(r))$, $r \rightarrow 1-0$, a sequence $(\theta_n)_{n \geq 0}$ satisfy condition (16), where $\varphi \in \mathcal{L}$ and*

$$\gamma(\varphi) = \overline{\lim}_{n \rightarrow +\infty} \frac{1}{\ln n} \ln \frac{\theta_n}{\theta_{n+1} - \theta_n} \leq \delta \in [0, 1). \quad (23)$$

Then for all analytic functions f_t there exists a set $E(\delta, t) \subset (0, 1)$ such that $h\text{-meas}(E(\delta, t)) < +\infty$ and almost surely for $t \in \mathbb{R}$

$$\overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) \leq \frac{1 + 2\delta}{4 + 2\delta}.$$

Proof. It follows from (22) that

$$\begin{aligned} & \ln \left(\varphi^{1/2} \left(\frac{h^{3/2}(r) \ln^{5/4} \{h(r) \mu_f(r)\} \ln_2^{1+\varepsilon} \{h(r) \mu_f(r)\}}{v(h^2(r) \ln \{h(r) \mu_f(r)\})} \right) \right) \leq \\ & \leq \left(\frac{3\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) 2 \ln h(r) + \left(\frac{5\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) \ln_2 \{h(r) \mu_f(r)\} \leq \\ & \leq \left(\frac{3\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) 2 \ln h(r) \leq \left(\frac{3\delta}{8} - \frac{\delta\alpha}{2} + o(1) \right) (2 \ln h(r) + \ln_2 \{h(r) \mu_f(r)\}). \end{aligned}$$

From the equation $\alpha = \frac{3\delta}{8} - \frac{\delta\alpha}{2}$ we determine $\alpha = \frac{3\delta}{4(2+\delta)}$ and

$$\begin{aligned} \overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) &= \overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \frac{\ln M_f(r, t) - \ln \mu_f(r)}{2 \ln h(r) + \ln_2 \{h(r) \mu_f(r)\}} \leq \\ &\leq \frac{1}{4} + \alpha = \frac{1}{4} + \frac{3\delta}{4(2+\delta)} = \frac{1 + 2\delta}{4 + 2\delta}. \end{aligned}$$

□

4 Baire's categories and Wiman-Valiron's type inequality for analytic functions

Let $h \in H$ and $\theta = (\theta_n)_{n \geq 0}$ be a fixed sequence satisfying condition (16), such that $\gamma(\varphi) \leq \delta$. Similarly to [11], we define the following sets

$$\begin{aligned} F_{1h}(f, \theta, E) &= \left\{ t \in \mathbb{R} : \overline{\lim}_{\substack{r \rightarrow 1-0 \\ r \notin E}} \Delta_h(r, f_t) \leq \frac{1+3\delta}{4+2\delta} \right\} \\ F_{2h}(f, \theta) &= \left\{ t \in \mathbb{R} : \left(\forall \eta > \frac{1+3\delta}{4+2\delta} \right) [h\text{-meas}(E(\eta, f_t, h)) < +\infty] \right\} \\ F_{3h}(f, \theta) &= \left\{ t \in \mathbb{R} : \underline{\lim}_{r \rightarrow 1-0} \Delta_h(r, f_t) \leq \frac{1+3\delta}{4+2\delta} \right\}, \\ F_{4h}(f, \theta) &= \left\{ t \in \mathbb{R} : \underline{\lim}_{r \rightarrow 1-0} \Delta_h(r, f_t) \leq \frac{1+2\delta}{4+2\delta} \right\}. \end{aligned}$$

By Corollary 3.5 we conclude that for analytic functions in \mathbb{D} there exists a set $E(f)$ of finite h -measure such that the set $F_{1h}(f, \theta)$ is “large” in the sense of Lebesgue measure. Therefore, we obtain some information on the sets $F_{2h}(f, \theta)$, $F_{3h}(f, \theta)$.

Similarly to [11], the following question arises naturally: *does there exist a set $E = E(f)$ of the finite h -measure such that the set $F_{1h}(f, \theta, E)$ is residual in \mathbb{R} for every analytic function f ?*

We recall that a set $B \subset \mathbb{R}$ is called residual in \mathbb{R} , if its complement $\overline{B} = \mathbb{R} \setminus B$ is a set of the first Baire category in \mathbb{R} . It is clear, that if the answer to the question is affirmative, then the sets $F_{2h}(f, \theta)$, $F_{3h}(f, \theta)$ are residual in \mathbb{R} . However for some analytic functions the set $F_{1h}(f, \theta, E)$ is a set of the first Baire category (see similar assertion for the entire function $f(z) = e^z$ in [11]). It follows from the following theorem.

Theorem 4.1. *Let a sequence $(\theta_n)_{n \geq 0}$ satisfy condition (4), $f(z) = \sum_{n=0}^{+\infty} e^{n\varepsilon} z^n$, $\varepsilon \in (0, 1)$, and $h(r) = (1-r)^{-1}$. Then there exists a constant $C = C(\theta, \varepsilon) > 0$ such that for all sequences $(r_n)_{n \geq 0}$ increasing to 1 the set*

$$F_3 = \left\{ t \in \mathbb{R} : \overline{\lim}_{n \rightarrow +\infty} \frac{M_{f_t}(r_n)}{h(r_n)\mu_f(r_n) \ln^{1/2}\{h(r_n)\mu_f(r_n)\}} \leq C \right\}$$

is a set of the first Baire category.

We need the following lemma from [12].

Lemma 4.2 ([12]). *For every $q > 1$ there exist positive constants $A = A(q)$ and $B = B(q)$ such that for each interval $I \subset \mathbb{R}$ and every trigonometrical*

polynomial $Q(t) = \sum_{n=1}^N c_n e^{i\lambda_n t}$, $0 < \lambda_1 < \lambda_2 < \dots < \lambda_N$, for which $|I| \geq \frac{B}{\lambda_1} > 0$ and $\frac{\lambda_{n+1}}{\lambda_n} \geq q$, $1 \leq n \leq N-1$, there exists a point $t_0 \in I$ such that

$$\operatorname{Re} Q(t_0) \geq A \sum_{n=1}^N |c_n|.$$

Proof of Theorem 4.1. For the function $f(z) = \sum_{n=1}^{+\infty} \exp\{n^\varepsilon\} z^n$, $\varepsilon \in (0, 1)$ (see [3]) there exists $C_0(\varepsilon) \in (0, 1)$ such that we have

$$C_0^{-1}(\varepsilon) \frac{\mu_f(r)}{1-r} \geq \frac{M_f(r)}{\sqrt{\ln M_f(r)}} \geq C_0(\varepsilon) \frac{\mu_f(r)}{1-r}, \quad r \rightarrow 1-0.$$

Then we obtain as $r \rightarrow 1-0$

$$\begin{aligned} \ln M_f(r) - \frac{1}{2} \ln_2 M_f(r) &\geq \ln C_0(\varepsilon) + \ln \frac{\mu_f(r)}{1-r}, \quad \ln M_f(r) \geq \ln \frac{\mu_f(r)}{1-r}, \\ M_f(r) &\geq C_0(\varepsilon) \frac{\mu_f(r)}{1-r} \ln^{1/2} \frac{\mu_f(r)}{1-r}. \end{aligned} \quad (24)$$

Let $(r_n)_{n \geq 0}$ be some sequence increasing to 1. We put

$$q = \inf\{\theta_{n+1}/\theta_n : n \geq 0\} > 1,$$

$A = A(q)$ and $B = B(q)$ are the constants from Lemma 4.2, $C(\varepsilon) = AC_0(\varepsilon)$. We consider a sequence of numbers $(C_n(\varepsilon))_{n \geq 0}$ increasing to $C(\varepsilon)$. Define the set

$$F_{mk} = \left\{ t \in \mathbb{R} : (\forall l \geq k) \left[M_{f_l}(r_l) \leq C_m(\varepsilon) \frac{\mu_f(r_l)}{1-r_l} \ln^{1/2} \frac{\mu_f(r_l)}{1-r_l} \right] \right\}.$$

where the integers $k \geq 0$, $m \geq 0$ are fixed.

For fixed $r \in (0, 1)$ we consider the function

$$\alpha(t, \varphi) = \left| \sum_{n=0}^{+\infty} \exp\{i\theta_n t + n^\varepsilon + in\varphi\} r^n \right|,$$

which is continuous in \mathbb{R}^2 and periodic in the variables t and φ . Then the function $\beta(t) = \max_\varphi \alpha(t, \varphi) = M_f(r, t)$ is continuous at every point $t \in \mathbb{R}$. We remark, that the set F_{mk} is closed in \mathbb{R} .

Now we prove that the set $\overline{F_{mk}}$ is everywhere dense. Consider an arbitrary interval $I \subset \mathbb{R}$, $|I| > 0$ and show that it contains some point t_0 from the set $\overline{F_{mk}}$.

Let us choose $p \geq 1, \delta > 0$ such that

$$|I| \geq \frac{B}{\theta_p}, \quad 1 - 2\delta > \sqrt{\frac{C_m(\varepsilon)}{C(\varepsilon)}}. \quad (25)$$

Using (24), we may define

$$x_1 = x_1(\varepsilon) = \inf \left\{ r \in (0, 1) : \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r^n \geq (1 - 2\delta) C_0(\varepsilon) \frac{\mu_f(r)}{1-r} \ln^{1/2} \frac{\mu_f(r)}{1-r} \right\},$$

$$x_2 = x_2(\varepsilon) = \inf \left\{ r \in (0, 1) : \sum_{n=0}^p \exp\{n^\varepsilon\} r^n \leq \frac{A}{A+1} \delta \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r^n \right\}.$$

Now choose integers $l \geq k$ and $s > p$ such that the following inequalities

$$r_l > \max\{x_1, x_2\}, \quad \sum_{n=s+1}^{+\infty} \exp\{n^\varepsilon\} r_l^n \leq \frac{A}{A+1} \delta \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r_l^n \quad (26)$$

hold.

By Lemma 2.2 there exists a point t_0 in the interval I such that

$$\operatorname{Re} \left(\sum_{n=p}^s e^{i\theta_n t_0 + n^\varepsilon} r_l^n \right) \geq A \sum_{n=p}^s \exp\{n^\varepsilon\} r_l^n. \quad (27)$$

Using the definitions of x_1, x_2 from (24)–(27) we deduce

$$\begin{aligned} M_f(r_l, t_0) &= \max_{\varphi} |f_{t_0}(r_l e^{i\varphi})| \geq \\ &\geq |f_{t_0}(r_l)| \geq \operatorname{Re} f_{t_0}(r_l) \geq \operatorname{Re} \left(\sum_{n=p}^s \exp\{i\theta_n t_0 + n^\varepsilon\} r_l^n \right) - \\ &- \sum_{n \notin [p, s]} \exp\{n^\varepsilon\} r_l^n \geq A \sum_{n=p}^s \exp\{n^\varepsilon\} r_l^n - \sum_{n \notin [p, s]} \exp\{n^\varepsilon\} r_l^n = \\ &= A \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r_l^n - (1+A) \sum_{n \notin [p, s]} \exp\{n^\varepsilon\} r_l^n \geq \\ &\geq A \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r_l^n - (1+A) \frac{2A}{1+A} \delta \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r_l^n = \\ &= A(1-2\delta) \sum_{n=0}^{+\infty} \exp\{n^\varepsilon\} r_l^n \geq \frac{C(\varepsilon)}{C_0(\varepsilon)} (1-2\delta)^2 C_0(\varepsilon) \frac{\mu_f(r_l)}{1-r_l} \ln^{1/2} \frac{\mu_f(r_l)}{1-r_l} \geq \\ &\geq C_m(\varepsilon) \frac{\mu_f(r_l)}{1-r_l} \ln^{1/2} \frac{\mu_f(r_l)}{1-r_l}. \end{aligned}$$

Therefore, $t_0 \in \overline{F_{mk}}$.

Since the set F_{mk} is closed in \mathbb{R} and its complement $\overline{F_{mk}}$ is everywhere dense, the set F_{mk} is nowhere dense. Hence

$$F_3 = \bigcup_{m=0}^{+\infty} \bigcup_{k=0}^{+\infty} F_{mk}$$

is a set of the first Baire category. \square

Theorem 4.3. *If a sequence $(\theta_n)_{n \geq 0}$ satisfies condition (16) and $h(r) = \frac{1}{1-r}$, then for every analytic function f the set $F_{3h}(f, \theta)$ is residual in \mathbb{R} .*

Proof. Let f be an arbitrary analytic function in \mathbb{D} . We consider the sequence $(c_n)_{n \geq 0}$ such that

$$c_n \downarrow \frac{1+3\delta}{4+2\delta}, \quad n \rightarrow +\infty.$$

Fix integers $m \geq 0$, $k \geq 0$ and define the set

$$G_{mk} = \left\{ t \in \mathbb{R} : M_f(r, t) \geq \frac{\mu_f(r)}{(1-r)^{c_m}} \ln^{c_m} \frac{\mu_f(r)}{1-r}, \quad \forall r > 1 - \frac{1}{k+1} \right\}.$$

As it has been proved above, for every fixed $r \in (0, 1)$ the function $\beta(t) = M_f(r, t)$ is continuous at every point $t_0 \in \mathbb{R}$. Then the set G_{mk} is closed in \mathbb{R} . By Corollary 3.5 the set $\overline{G_{mk}}$ is everywhere dense. Therefore, G_{mk} is nowhere dense and

$$G = \bigcup_{m=0}^{+\infty} \bigcup_{k=0}^{+\infty} G_{mk}$$

is a set of the first Baire category. So, $F_{3h}(f, \theta) = \overline{G}$ is residual in \mathbb{R} . \square

Theorem 4.4. *If a sequence $(\theta_n)_{n \geq 0}$ satisfies condition (16) and $h(r) = \frac{1}{1-r}$, then for all analytic functions f such that $\ln_2 \mu_f(r) = o(\ln(1-r))$, $r \rightarrow 1-0$, the set $F_{4h}(f, \theta)$ is residual in \mathbb{R} .*

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